Quantum time scales and the classical limit: Analytic results for some simple systems

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We set up a semiclassical approximation which helps us clarify by means of several simple examples the rich variety of time scale in the quantum domain. The underlying structure of quantum and classical mechanics is so completly different that it is naive to expect to reach a classical regime by counting powers of the quantum scale $\hbar$. We show although it is possible to define a time scale for nonclassical phenomena, but it is impossible to characterize quantum dynamics through a unique time scale, such as Ehrenfest's time. We use simple systems to critically discuss and illustrate these features of the quantum-classical limit.

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I. INTRODUCTION

The dynamical behavior of point particles as described by Newtonian mechanics is considerably altered when relativistic effects are included. The same is valid when one goes over to a nonrelativistic quantum description. In the first case (special relativity) there exists an obvious parameter, light velocity, which, when compared to the velocities of the problem in question, naturally shows the way to recover Newtonian dynamics, namely an expansion power of v/c. In the second case (nonrelativistic quantum mechanics) the situation is far more complicated. Although, since the early days of Quantum Mechanics it became clear that quantum phenomena bear the imprint of $\hbar$, there is a new ingredient that is not present in the first case: the underlying quantum kinematics is essentially different from that of Newtonian mechanics whereas in special relativity this is not the case. Quantum and classical mechanics are two essentially different theories both from the point of view of their underlying kinematical construction and their dynamics. If quantum mechanics should, as many of us expect, possess a universal character, it should be possible to derive classical mechanics from the corresponding quantum system. Many attempts in this direction have been put forward. One set of papers that deal with this question are [1–3]. In these references the authors assume the classical state to be a classical probability distribution by comparing with the corresponding quantum evolution. They show that Ehrenfest's theorem is neither necessary nor sufficient to characterize the classical regime in quantum theory. Therefore Ehrenfest's theorem does not define the conditions for classical behavior. The time during which the first moments of the two distributions coincide is usually called Liouville time. The purpose of the present work is somewhat complementary to the papers discussed above. We consider the case of a Newtonian particle whose state is given by a point in phase space. We show that the quantum regime of these point particles present several nonclassical phenomena, characterized by different time scales. We derive and critically discuss all these time scales. It is worthwhile mentioning here that the spreading of wave packets can be naturally accommodated in a classical context if one is talking about classical probabilities as representative of the state of the particles. A precise discussion of this point can be found in Ref. [1]. In the present contribution we confine ourselves to a comparison between Newtonian particles and their quantum counterpart when this exists.

In order to obtain time scales for the evolution of different observables and compare with their classical limit wherever this can be defined, we construct a self-consistent expansion for the wave function of the system around a time dependent coherent state (or product of coherent states), whose dynamics is completely given by the underlying classical equations of motion. In this way all next-to-leading order terms in the expansion are essentially of quantum character. For example, the next-to-leading order already introduces an essentially quantum ingredient, i.e., a linear superposition of quantum states. Naturally, such correction will affect the time evolution of observables to a lesser or greater extent, depending on how sensitive the particular observable (and initial condition) is to this correction.

This paper is organized as follows: in Sec. II we present the method, which is a generalization of Ref. [4] in order to include one degree of freedom systems. As an illustration of the method, quantum and classical time scales are presented for the one-dimensional quartic oscillator. Section III contains a detailed analysis of the dynamics of the two-dimensional quartic oscillators, where typical entanglement time scales are also obtained and compared with the Ehrenfest time related to the time evolution of positions and momenta. We show that both depend on $\hbar$, they are quite different. Within the validity of the approximation we show that Ehrenfest's time is usually longer than entanglement time. The reason for this becomes apparent in our approximation scheme, where it is easy to see that the first nonvanishing correction to the Wigner function brings in entanglement,

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and measures of coherence content usually are more sensitive to this aspect than mean values of one-body observables.

II. THE APPROXIMATION SCHEME AND THE ONE-DIMENSIONAL QUARTIC OSCILLATOR

Shortly after the birth of lchanics, Schrödinger, proposed the idea of a coherent state [5]. His motivation was the construction of wave packets whose centers follow the motion of a classical point particle, while retaining their shapes, we follow his idea closely. More recently another important development which concerns coherent states was the recognition that coherent states are intimately connected to the dynamical group of the physical problem. For example, when one includes only creation, annihilation, identity, and number conserving operator as generators, the system possesses the Heisenberg-Weyl $H(4)$ dynamical group. There is then a natural correspondence with known geometrical properties [6]. The physical interpretation of the parameter space structure for the system, and the equations of motion of the coherent state parameters precisely correspond to the classical ones. For the above discussed reason we construct a semiclassical expansion of the dynamical equation of a quantum system, where the zeroth-order approximation is the description with the minimum of the quantum ingredients and all classical ones. Typical quantum effects as spreading of wave packets and superpositions are all contained only in higher orders. We set up the approximation schemes for one degree of freedom systems (closed or not). The generalization is straightforward.

Let us consider a classical one degree of freedom Hamiltonian of the form

$$H_{\mathrm{cls}}=\frac{p^2}{2m}+V(q), \tag{1}$$

where p stands for the particle momentum and q for its position. We perform the following change of variables:

$$p=\frac{\alpha-\alpha^*}{i\sqrt{\dfrac{2}{m\omega\hbar}}},\quad q=\frac{\alpha+\alpha^*}{\sqrt{\dfrac{2m\omega}{\hbar}}}, \tag{2}$$

where $\omega=\sqrt{k/m}$ and $k=\partial^2V(q)/\partial q^2|_{q=0}$.

The Hamiltonian can then be rewritten as

$$H_{\mathrm{cls}}=\hbar\omega\alpha^*\alpha+U(\alpha^*\alpha) \tag{3}$$

with $U(\alpha^*\alpha)=V(q)-[k(\alpha+\alpha^*)/\sqrt{2m\omega/\hbar}]^2$.

It is now possible to write H_{cls} in the form of a Taylor expansion such

$$H_{\mathrm{cls}}=\hbar\omega\alpha^*\alpha+\sum_{m,n}A_{m,n}(\alpha^*)^m\alpha^n,$$

where $A_{1,1}=0$.

The classical equations of motions read

$$\frac{d}{dt}\alpha=\frac{1}{i\hbar}\frac{\partial H_{\mathrm{cls}}}{\partial\alpha^*}=-i\omega\alpha-\frac{i}{\hbar}\sum_{m,n}mA_{m,n}(\alpha^*)^{m-1}\alpha^n, \tag{4}$$

$$\frac{d}{dt}\alpha^*=-\frac{1}{i\hbar}\frac{\partial H_{\mathrm{cls}}}{\partial\alpha}=i\omega\alpha^*+\frac{i}{\hbar}\sum_{m,n}nA_{m,n}(\alpha^*)^m\alpha^{n-1}. \tag{5}$$

One possible quantum realization of H_{cls} is given by

$$\hat{H}_q=\hbar\omega\hat{a}^\dagger\hat{a}+\sum_{m,n}A_{m,n}(\hat{a}^\dagger)^m\hat{a}^n.$$

Note that $\langle\alpha|\hat{H}_q|\alpha\rangle=H_{\mathrm{cls}}$, if $|\alpha\rangle$ is a coherent field state.

We will construct an expansion around a quantum operator $H_{\mathrm{sc}}(\alpha(t))$ which possess the following features (a) $\langle\alpha|H_{\mathrm{sc}}(\alpha(t))|\alpha\rangle=H_{\mathrm{cls}}$; (b) $i\hbar\partial/\partial t|\alpha\rangle=\hat{H}_{\mathrm{sc}}(\alpha(t))|\alpha\rangle$; (c) All expectation values of point classical observables will be precisely reproduced by $\hat{H}_{\mathrm{sc}}(\alpha(t))$ (including variances). The construction of the semiclassical Hamiltonian is such that it is completely determined by the classical equations of motion $\alpha_{\mathrm{cl}}(t)$ and its operator form is the following

$$\hat{H}_{\mathrm{sc}}=\hbar\omega\hat{a}^\dagger\hat{a}+B_0\hat{a}^\dagger\hat{a}+B_1\hat{a}^\dagger+B_2\hat{a}. \tag{6}$$

We show next how to construct such an operator. We start by showing that if $|\alpha(t)\rangle$ is a time dependent solution of Eq. (6), i.e.,

$$i\hbar\frac{\partial}{\partial t}|\alpha\rangle=\hat{H}_{\mathrm{sc}}(\alpha(t))|\alpha\rangle,$$

it is necessary to verify that

$$\frac{d}{dt}\alpha=-i\left(\omega+\frac{B_0}{\hbar}\right)\alpha-\frac{i}{\hbar}B_1, \tag{7}$$

$$\frac{d}{dt}\alpha^*=i\left(\omega+\frac{B_0^*}{\hbar}\right)\alpha^*+\frac{i}{\hbar}B_2. \tag{8}$$

Next we further demand that Eqs. (7) and (8) correspond precisely to Eqs. (4) and (5), thus guaranteeing that the semiclassical operator will be completely and solely (apart from a phase) given by the classical equations of motions. Thus we require

$$-i\left(\frac{B_0}{\hbar}\right)\alpha-\frac{i}{\hbar}B_1=-\frac{i}{\hbar}\sum_{m,n}mA_{m,n}(\alpha^*)^{m-1}\alpha^n, \tag{9}$$

$$i\left(\frac{B_0^*}{\hbar}\right)\alpha^*+\frac{i}{\hbar}B_2=\frac{i}{\hbar}\sum_{m,n}nA_{m,n}(\alpha^*)^m\alpha^{n-1}. \tag{10}$$

From the above equations we see that there exist several $\hat{H}_{\mathrm{sc}}$ satisfying these conditions. In order to have uniqueness, we demand further that the operators $\hat{a}^\dagger\hat{a}$, $\hat{a}^\dagger$, and $\hat{a}$ obey an equation which is similar in structure to the classical one:

$$\frac{d}{dt}\hat{a}=\left(-i\omega-\frac{i}{\hbar}\sum_{m,n} m\,A_{m,n}(\alpha^*)^{m-1}\alpha^{n-1}\right)\hat{a},$$

$$\frac{d}{dt}\hat{a}^\dagger=\left(i\omega+\frac{i}{\hbar}\sum_{m,n} n\,A_{m,n}(\alpha^*)^{m-1}\alpha^{n-1}\right)\hat{a}^\dagger.$$

The semiclassical Hamiltonian that satisfies this condition is

$$\hat{H}_{\rm sc}=\hbar\omega\hat{a}^\dagger\hat{a}+\sum_{m\neq 0} m\,A_{m,m}(\alpha^*)^{m-1}\alpha^{m-1}\hat{a}^\dagger\hat{a}$$
$$+\sum_{m\neq n} m\,A_{m,n}(\alpha^*)^{m-1}\alpha^{n}\,\hat{a}^\dagger$$
$$+\sum_{m,n} n\,A_{m,n}(\alpha^*)^{m}\alpha^{n-1}\hat{a}.$$

Note now that $\langle\alpha|\hat{H}_{\rm sc}|\alpha\rangle\neq H_{\rm cls}$, but this problem is simple to solve. We only have to add constant terms, which contributes an overall phase for the time evolved state $|\alpha(t)\rangle$. Finally, we write

$$\hat{H}_{\rm sc}(\alpha(t))=\hbar\omega\hat{a}^\dagger\hat{a}+\sum_{m\neq 0} m\,A_{m,m}(\alpha^*)^{m-1}$$
$$\times\alpha^{m-1}(\hat{a}^\dagger\hat{a}-\alpha^*\alpha)+\sum_{m,n} A_{m,n}(\alpha^*)^{m}\alpha^{n}$$
$$+\sum_{m\neq n} m\,A_{m,n}(\alpha^*)^{m-1}\alpha^{n}(\hat{a}^\dagger-\alpha^*)$$
$$+\sum_{m,n} n\,A_{m,n}(\alpha^*)^{m}\alpha^{n-1}(\hat{a}-\alpha). \tag{11}$$

The full quantum time evolution can be formally written ass

$$|\Psi(t)\rangle=\hat{U}_{\rm sc}(t)\left(1+\frac{1}{i\hbar}\int_0^t dt_1\Delta_s(t_1)\right.$$
$$\left.+\frac{1}{(i\hbar)^2}\int_0^t\int_0^{t_1}dt_2dt_2\Delta_s(t_1)\Delta_s(t_2)+\cdots\right)|\Psi(0)\rangle, \tag{12}$$

where $\hat{U}_{\rm sc}(t)=\exp^{-i/\hbar\int_0^t\hat{H}_{\rm sc}(t')dt'}$ and

$$\Delta_s=\hat{U}_{\rm sc}^\dagger(t)\,\delta\hat{U}_{\rm sc}(t)\quad\text{and}\quad \delta(\alpha(t))=\hat{H}_q-\hat{H}_{\rm sc}(\alpha(t)). \tag{13}$$

A similar expansion for the Heisenberg operator can be written as

$$i\hbar\frac{d}{dt}\hat{O}=[\hat{O},\hat{H}].$$

Now writing $\hat{H}=\hat{H}_{\rm sc}+\delta$, we have

$$i\hbar\frac{d}{dt}\hat{O}=[\hat{O},\hat{H}_{\rm sc}]+[\hat{O},\delta].$$

In the interaction picture representation

$$\hat{O}_I=\exp\left\{-\frac{i}{\hbar}\int_0^t\hat{H}_{\rm sc}(t')dt'\right\}\hat{O}\exp\left\{+\frac{i}{\hbar}\int_0^t\hat{H}_{\rm sc}(t')dt'\right\}. \tag{14}$$

We thus have

$$\Delta_H(t)=\exp\left\{-\frac{i}{\hbar}\int_0^t\hat{H}_{\rm sc}(t')dt'\right\}$$
$$\times\delta\left(\exp\left\{+\frac{i}{\hbar}\int_0^t\hat{H}_{\rm sc}(t')dt'\right\}\right)$$
$$=\hat{U}_{\rm sc}(t)\,\delta\hat{U}_{\rm sc}^\dagger(t) \tag{15}$$

and hence

$$i\hbar\frac{d}{dt}\hat{O}_I=[\hat{O}_I,\Delta_H]. \tag{16}$$

Iterating Eq. (16), we get

$$\hat{O}_{\rm I}(t)=\hat{O}_{\rm II}(t)-\frac{i}{\hbar}\int_0^t[\hat{O}_{\rm II}(t),\tilde{\Delta}_H(t_1,t)]dt_1-\left(\frac{1}{\hbar}\right)^2$$
$$\times\int_0^t\int_0^{t_1}[[\hat{O}_{\rm II}(t),\tilde{\Delta}_H(t_2,t)],\tilde{\Delta}_H(t_1,t)]dt_2dt_1$$
$$+i\left(\frac{1}{\hbar}\right)^3\int_0^t\int_0^{t_1}\int_0^{t_2}[[[\hat{O}_{\rm II}(t),\tilde{\Delta}_H(t_3,t)],$$
$$\tilde{\Delta}_H(t_2,t)],\tilde{\Delta}_H(t_1,t)]dt_3dt_2dt_1+\cdots, \tag{17}$$

where

$$\hat{O}_{\rm II}(t)=\hat{U}_{\rm sc}^\dagger(t)\hat{O}_{\rm I}(0)\hat{U}_{\rm sc}(t)$$

and

$$\tilde{\Delta}_H(t_i,t)=\hat{U}_{\rm sc}^\dagger(t)\hat{U}_{\rm sc}(t_i)\,\delta(\alpha(t_i))\hat{U}_{\rm sc}^\dagger(t_i)\hat{U}_{\rm sc}(t).$$

This expansion is not restricted to particle systems, but can be used for spin systems or any system for which a corresponding coherent representation exists. A generalization of this scheme to nonautonomous systems is straightforward. An eventually classically chaotic dynamics is contained in the zeroth order approximation of this expansion. It is specially well suited for assessing the characteristic times for the manifestation of quantum effects in any observable.

A. An example: The quartic oscillator

Let us consider the following classical Hamiltonian:

$$H_{\rm cls}=\frac{p^2}{2m}+\frac{kq^2}{2}+\lambda\frac{m}{k}\left(\frac{p^2}{2m}+\frac{kq^2}{2}\right)^2. \tag{18}$$

This system has been considered by several authors in the context of classical limit of observables [7–11], and in spite of its simplicity it has been used to explain some experimental results, see Ref. [12]. In what follows we present a critical review of these calculations, including also the time evolution of the corresponding state.

Using the previous prescriptions, we can also write the $H_{\rm cls}$ as

$$H_{\rm cls}=\hbar\omega\alpha^*\alpha+\lambda\hbar^2(\alpha^*)^2\alpha^2. \tag{19}$$

1. The classical solution

The equations of motion of the model read

$$\frac{d}{dt}\alpha=\frac{1}{i\hbar}\frac{\partial H_{\rm cls}}{\partial\alpha^*}=-i\omega\alpha-i2\lambda\hbar\alpha^*\alpha^2,$$

$$\frac{d}{dt}\alpha^*=-\frac{1}{i\hbar}\frac{\partial H_{\rm cls}}{\partial\alpha}=i\omega\alpha^*+i2\lambda\hbar(\alpha^*)^2\alpha,$$

and the solution is then

$$\alpha(t)=\alpha(0)\exp\{[-i\omega-i2\lambda\hbar|\alpha(0)|^2]t\}. \tag{20}$$

2. The exact quantum solution

According to the definition of $\hat{H}_q$, we have

$$\hat{H}_q=\hbar\omega\hat{a}^\dagger\hat{a}+\lambda\hbar^2(\hat{a}^\dagger)^2\hat{a}^2. \tag{21}$$

Using a coherent state as an initial condition we wish to evaluate

$$|\Psi(t)\rangle=e^{-i\hat{H}_q t/\hbar}|\alpha(0)\rangle.$$

In order to compare with the classical time evolution, we evaluate $\langle\hat{a}\rangle(t)$, given by

$$\langle\hat{a}\rangle(t)=\langle\alpha(0)|\hat{a}(t)|\alpha(0)\rangle.$$

As can be easily checked, the full quantum solution is

$$\langle\hat{a}\rangle(t)=\alpha(0)e^{-i\omega t}e^{-|\alpha(0)|^2[1-\exp(-2it\lambda\hbar)]}. \tag{22}$$

3. The semiclassical solution

In order to get a feel of for the approximation we developed we next use it to obtain a semiclassical approximation for $\langle\hat{a}\rangle(t)$. For this propose we write

$$\hat{H}_q=\sum_{m=1}^{2}A_m(\hat{a}^\dagger)^m\hat{a}^m,$$

where $A_1=\hbar\omega$, $A_2=\lambda\hbar^2$. According to the prescriptions of this section beginning we get

$$\hat{H}_{\rm sc}=\hbar\omega\hat{a}^\dagger\hat{a}+2\lambda\hbar^2\alpha^*\alpha\hat{a}^\dagger\hat{a}-\lambda\hbar^2(\alpha^*)^2\alpha^2.$$

According to its definition, Eq. (13), we have

$$\delta(\alpha(t))=-2\lambda\hbar^2\alpha^*\alpha\hat{a}^\dagger\hat{a}+\lambda\hbar^2(\hat{a}^\dagger)^2\hat{a}^2+\lambda\hbar^2(\alpha^*)^2\alpha^2.$$

Since $[\delta,\hat{H}_{\rm sc}]=0\Rightarrow\delta=\Delta_H$, then

$$\Delta_H=-2\lambda\hbar^2\alpha^*\alpha\hat{a}^\dagger\hat{a}+\lambda\hbar^2(\hat{a}^\dagger)^2\hat{a}^2=\mu\hat{a}^\dagger\hat{a}+\eta(\hat{a}^\dagger)^2\hat{a}^2.$$

The first terms of the semiclassical approximation for the expectation value of $\langle\hat{a}\rangle(t)$ are

$$\langle\hat{a}^0\rangle=\alpha_{\rm cl}(t),$$

$$\langle\hat{a}^1\rangle=\frac{t}{i\hbar}\langle\alpha(t)|[\hat{a},\Delta_H]|\alpha(t)\rangle=0,$$

$$\langle\hat{a}^2\rangle=-\frac{t^2}{2\hbar^2}\langle\alpha(t)t|[\mu+2\eta(\hat{a}^\dagger)\hat{a}]^2\hat{a}|\alpha(t)\rangle$$
$$=-2t^2\alpha(t)\hbar^2\lambda^2|\alpha(0)|^2,$$

$$\langle\hat{a}^3\rangle=-\frac{t^3}{i6\hbar^3}\langle\alpha(t)|[\mu+2\eta(\hat{a}^\dagger)\hat{a}]^3\hat{a}|\alpha(t)\rangle$$
$$=-\frac{8}{6i}t^3\alpha(t)\hbar^3\lambda^3|\alpha(0)|^2.$$

It is easy to see that the summation to all orders will yield the exact result. The zeroth order contribution corresponds to the classical evolution. The next-to-leading order allows us to determine the Ehrenfest time

$$t_E\sim[\hbar\lambda|\alpha(0)|\sqrt{2}]^{-1}. \tag{23}$$

A few comments on the above expression are in order. (a) The obtained expression for Ehrenfest's time explicitly depends on some kinematical ingredients e.g., the width of the initial wave packet, which, of course, depends on $\hbar$. This $\hbar$ dependence is therefore of no fundamental significance.

(b) The physical agent that produces the observed deviation from the classical point particle behavior is the nonlinearity reflected in the constant λ, which will tend to distort the initial wave packet. This very same effect, however, is also present in a classical probabilistic description. Therefore in this sense it is at least ambiguous to call Ehrenfest's time a quantum time scale in the present context. This has been numerically verified [13] and an analytical expression for expectation values with the classical statistical wave packet is presently under investigation.

At this point and for this particular observable it is meaningful to define a classical limit in a rigorous way. The amplitude of the coherent state $|\alpha(0)|^2$ is directly related to the ratio of the classical action and $\hbar$. This ratio may be used to define a classical limit. Note then that upon substitution of $\hbar|\alpha(0)|^2=S_{\rm cl}$, were $S_{\rm cl}$ is the classical action of the system, one gets the usually obtained Ehrenfest time were $\hbar$ appears and was interpreted in Refs. [7,8] as a quantum correction.

Note however that the result just obtained originated from a perturbative expansion (in the nonlinearity governed by the parameter λ). The expansion parameter to be considered is $\lambda\hbar|\alpha|^2 \ll 1$ [see Eq. (22)]. Within this limit, the above conclusions hold, as expected.

However, the same expansion while leading to the above result for the expectation value of an operator, shows different time scales for the Wigner function. In particular we show below that the characteristic time of the appearance of quantum effects has a completely different dependence of $\hbar$.

B. Wigner differential equation for the one-dimensional model

For the quartic one-dimensional oscillator we have the following exact equation for the density:

$$\dot{\rho} = -i[\underline{\omega}\hat{a}^\dagger\hat{a} + \lambda\hbar(\hat{a}^\dagger\hat{a})^2, \rho], \tag{24}$$

where $\underline{\omega} = \omega - \lambda\hbar$. Then the corresponding Wigner function obeys the following differential equation

$$\dot{W}(\eta) = -i(\tilde{\omega} + 2\hbar\lambda|\eta|^2)\left(\eta^*\frac{\partial}{\eta^*} - \eta\frac{\partial}{\partial\eta}\right)W(\eta) - \frac{i\lambda\hbar}{2}\left(\eta\frac{\partial^3}{\partial\eta^2\partial\eta^*} - 2\eta^*\frac{\partial^3}{\partial\eta^{*2}\partial\eta}\right)W(\eta).$$

For the semiclassical expansion of the Wigner function (zeroth order term) we find

$$\dot{W}_{\rm sc}(\eta) = -i\tilde{\omega}\left(\eta^*\frac{\partial}{\eta^*} - \eta\frac{\partial}{\partial\eta}\right)W_{\rm sc}(\eta),$$

where $\tilde{\omega} = \omega + 2\lambda\hbar|\alpha(t)|^2 = \omega + 2\lambda\hbar|\alpha_0|^2$. Now we proceed to derive the corrections to the Wigner function according to our approximation scheme. The semiclassical expansion for the density operator is given by

$$\begin{aligned}\rho(t) = \hat{U}_{\rm sc}(t)\Bigg\{&\rho(0) + \frac{i}{\hbar}\int_0^t[\rho(0), \Delta_s(t_1)]dt_1 \\ &- \left(\frac{1}{\hbar}\right)^2\int_0^t\int_0^{t_1}[[\rho(0),\Delta_s(t_2)],\Delta_s(t_1)]dt_2dt_1 \\ &- i\left(\frac{1}{\hbar}\right)^3\int_0^t\int_0^{t_1}\int_0^{t_2}[[[\rho(0),\Delta_s(t_3)],\Delta_s(t_2)],\Delta_s(t_1)] \\ &\times dt_3dt_2dt_1 + \cdots\Bigg\}\hat{U}_{\rm sc}^\dagger(t),\end{aligned} \tag{25}$$

were we have defined

$$\begin{aligned}\Delta_s &= \exp\left\{\frac{i}{\hbar}\int_0^t\hat{H}_{\rm sc}(t')dt'\right\}\Delta\exp\left\{-\frac{i}{\hbar}\int_0^t\hat{H}_{\rm sc}(t')dt'\right\} \\ &= \hat{U}_{\rm sc}^\dagger(t)\Delta\hat{U}_{\rm sc}(t).\end{aligned} \tag{26}$$

C. First correction to the Wigner function for the quartic model

From Eq. (25) we can write the density operator as

$$\rho(t) = \rho_{\rm sc}(t) + \rho^1(t) + \rho^2(t) + \cdots. \tag{27}$$

Then we have

$$W(t) = W_{\rm sc}(t) + W^1(t) + W^2(t) + \cdots. \tag{28}$$

The first correction is due to the first term of the expansion, that is

$$\begin{aligned}\rho^1(t) &= \hat{U}_{\rm sc}(t)\left\{\frac{i}{\hbar}\int_0^t[\rho(0),\Delta_s(t_1)]dt_1\right\}\hat{U}_{\rm sc}^\dagger(t) \\ &= \frac{i}{\hbar}t\hat{U}_{\rm sc}(t)[\rho(0),\Delta]\hat{U}_{\rm sc}^\dagger(t),\end{aligned} \tag{29}$$

where $\Delta = \mu\hat{a}^\dagger\hat{a} + 2\eta(\hat{a}^\dagger)^2\hat{a}^2$ and $\rho(0) = |\alpha_0\rangle\langle\alpha_0|$. Then after some algebra we find

$$\begin{aligned}\rho^1(t) = \frac{i}{\hbar}t\{&|\alpha_t\rangle\langle\alpha_t|[\mu\alpha_t^*\hat{a} + 2\eta(\alpha_t^*)^2\hat{a}^2] \\ &- [\mu\hat{a}^\dagger\alpha_t + 2\eta(\hat{a}^\dagger)^2\alpha_t^2]|\alpha_t\rangle\langle\alpha_t|\},\end{aligned} \tag{30}$$

where $\alpha(t)$ is given by Eq. (20). The first Wigner term is then

$$\begin{aligned}W^1(\nu) = \frac{-16\lambda\hbar}{\pi}t\exp\{&4{\rm Re}(\nu^*\alpha_t) - 2|\alpha_t|^2 - 2|\nu|^2\} \\ &\times{\rm Im}(2\nu^2(\alpha_t^*)^2 - 4|\alpha|^2\alpha_t^*\nu).\end{aligned} \tag{31}$$

The Wigner function of coherent state $|\alpha\rangle$ is

$$W_{sc}(\nu) = \frac{2}{\pi}\exp\{4{\rm Re}(\nu^*\alpha_t) - 2|\alpha_0|^2 - 2|\nu|^2\}. \tag{32}$$

Then for the first order in time, we may approximate $\alpha(t)$ as

$$\alpha(t) = \alpha(0)(1 - i\tilde{\omega}t), \tag{33}$$

where $\tilde{\omega} = \omega + 2\lambda\hbar|\alpha(0)|^2$.

Then the semiclassical Wigner is [14]

$$\begin{aligned}W_{\rm sc}(\nu) = \frac{2}{\pi}\exp\{&4\,{\rm Re}(\nu^*\alpha_0) - 2|\alpha_0|^2 - 2|\nu|^2\} \\ &\times[1 - 4t\tilde{\omega}\,{\rm Im}(\nu\alpha_0^*)].\end{aligned} \tag{34}$$

Note that the study of short times for the evolution of the state is far more complicated than that for expectation values as shows the discussion below. In particular the first nonzero correction is of first order while that for the expectation values is of second order.

The validly of Eq. (33) may depend on time and some combination of parameters. Since the Wigner function of a coherent state is always positive,

$$1-4t\tilde{\omega}\,\mathrm{Im}(\nu\alpha_0^*)\geqslant 0,\quad t_{1c}\leqslant[4\tilde{\omega}\,\mathrm{Im}(\nu\alpha_0^*)]^{-1}.$$

In order to arrive at a simple expression, let us study its short time behavior:

$$W(\nu)\simeq\frac{2}{\pi}\exp\{4\,\mathrm{Re}(\nu^*\alpha_0)-2|\alpha_0|^2-2|\nu|^2\}\times[1-4t\tilde{\omega}\,\mathrm{Im}(\nu\alpha_0^*)-8t\lambda\hbar\,\mathrm{Im}\{2\nu^2(\alpha_t^*)^2-4|\alpha|^2\alpha_t^*\nu\}].\tag{35}$$

Now we want to investigate if there are negative regions in the Wigner function. Therefore we must have

$$1-4t_{2c}\tilde{\omega}\,\mathrm{Im}(\nu\alpha_0^*)-8t_{2c}\lambda\hbar\,\mathrm{Im}(2\nu^2\alpha_t^{*2}-4|\alpha|^2\alpha_t^*\nu)<0,$$

or equivalently,

$$t_{2c}>\frac{1}{4\tilde{\omega}\,\mathrm{Im}(\nu\alpha_0^*)+8\lambda\hbar\,\mathrm{Im}(2\nu^2\alpha_t^{*2}-4|\alpha|^2\alpha_t^*\nu)}.$$

This is the characteristic time for the appearance of quantum effects. It is surely smaller than the ''Ehrenfest time'' [cf Eq. (23)]. This means that the operator chosen is ''blind'' to the particular correlations developed in the Wigner function for short times. Then if we want to see interference terms we should have $t_{2c}<t_{1c}$, so that

$$\frac{1}{4\tilde{\omega}\,\mathrm{Im}(\nu\alpha_0^*)+8\lambda\hbar\,\mathrm{Im}(2\nu^2\alpha_t^{*2}-4|\alpha|^2\alpha_t^*\nu)}\leqslant[4\tilde{\omega}\,\mathrm{Im}(\nu\alpha_0^*)]^{-1},\tag{36}$$

and hence

$$8\lambda\hbar\,\mathrm{Im}(2\nu^2\alpha_t^{*2}-4|\alpha|^2\alpha_t^*\nu)\geqslant 0.\tag{37}$$

To satisfy this, we may have

$$\mathrm{Im}(\nu)\geqslant 2\,\mathrm{Im}(\alpha).\tag{38}$$

Therefore, negative regions will develop for short times in phase space regions which satisfy the above condition, i.e., mainly away from the center of the Gaussian.

The characteristic time for these quantum developments is rather intricate and it is therefore hard to characterize it only in terms of $\hbar$. In Figs. 1 and 2 we show the zeroth- and first-order Wigner functions for the parameters indicated in the figure. Note that $\mathrm{Tr}(a\rho^1)=0$ showing thus unambiguously that the characteristic time of the expectation value is rather different than that of the state.

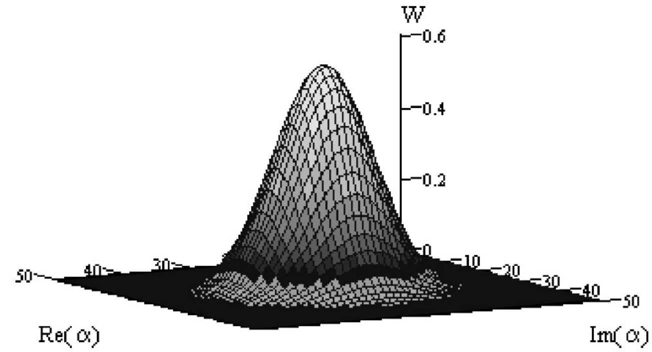

FIG. 1. First order of the Wigner function for the quartic one-dimensional oscillator. α is dimensionless, see its definition Eq. (2).

III. TWO COUPLED QUARTIC OSCILLATOR

In this section we discuss the time scale of another typical quantum process, entanglement and its relation to $\hbar$. For our purposes it suffices to consider a simple two-dimensional model, which is exactly soluble:

$$H_{\mathrm{cls}}=\frac{p_1^2}{2m}+\frac{kq_1^2}{2}+\frac{p_2^2}{2m}+\frac{kq_2^2}{2}+\lambda\frac{m}{k}\left(\frac{p_1^2}{2m}+\frac{kq_1^2}{2}+\frac{p_2^2}{2m}+\frac{kq_2^2}{2}\right)^2+\mu\hbar\,\mathrm{Re}\left[\left(i\sqrt{\frac{2}{m\omega\hbar}}p_1+\sqrt{\frac{2m\omega}{\hbar}}q_1\right)\left(i\sqrt{\frac{2}{m\omega\hbar}}p_2+\sqrt{\frac{2m\omega}{\hbar}}q_2\right)\right].$$

By making analogous substitution as in the preceding section we rewrite the classical Hamiltonian as

$$H_{\mathrm{cls}}=\hbar\omega(\alpha^*\alpha+\beta^*\beta)+\lambda\hbar^2[(\alpha^*)^2\alpha^2+(\beta^*)^2\beta^2+2\alpha^*\alpha\beta^*\beta]+\mu\hbar(\alpha^*\beta+\alpha\beta^*).\tag{39}$$

The classical solution is given by

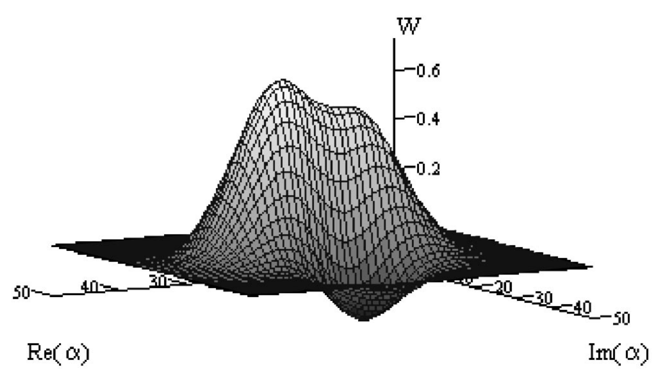

FIG. 2. Second order of the Wigner function for the quartic one-dimensional oscillator. α is dimensionless, see its definition Eq. (2).

$$\begin{bmatrix}\alpha(t)\\ \beta(t)\end{bmatrix}=\begin{bmatrix}\frac{1}{2}(e^{-i(\Omega+\mu)t}+e^{-i(\Omega-\mu)t}) & \frac{1}{2}(e^{-i(\Omega+\mu)t}-e^{-i(\Omega-\mu)t})\\ \frac{1}{2}(e^{-i(\Omega+\mu)t}-e^{-i(\Omega-\mu)t}) & \frac{1}{2}(e^{-i(\Omega+\mu)t}+e^{-i(\Omega-\mu)t})\end{bmatrix}\begin{bmatrix}\alpha_0\\ \beta_0\end{bmatrix}. \tag{40}$$

A. The semiclassical solution

Now $\hat{H}_q$ is

$$\hat{H}_q=\hbar\omega(\hat{a}^\dagger\hat{a}+\hat{b}^\dagger\hat{b})+\sum_{m,n=0}^{2}A_{m,n}(\hat{a}^\dagger)^m\hat{a}^m(\hat{b}^\dagger)^n\hat{b}^n$$
$$+\mu\hbar(\hat{a}^\dagger\hat{b}+\hat{a}\hat{b}^\dagger),$$

where $A_{10}=A_{01}=0$, $A_{20}=A_{02}=A_{11}/2=\lambda\hbar^2$ and $A_{22}=0$.

As before, we may write the semiclassical Hamiltonian as

$$\hat{H}_{\rm sc}=\hbar\Omega(\hat{a}^\dagger\hat{a}+\hat{b}^\dagger\hat{b})+\mu\hbar[\alpha^*\hat{b}+\alpha\hat{b}^\dagger+\hat{a}^\dagger\beta+\hat{a}\beta^*]$$
$$-\lambda\hbar^2[(\alpha^*)^2\alpha^2+(\beta^*)^2\beta^2+2\alpha^*\alpha\beta^*\beta]$$
$$-\mu\hbar(\alpha^*\beta+\alpha\beta^*).$$

According to Eq. (13) we get

$$\delta=\lambda\hbar^2[(\hat{a}^\dagger)^2\hat{a}^2+(\hat{b}^\dagger)^2\hat{b}^2+2\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b}]+\mu\hbar[\hat{a}\hat{b}^\dagger+\hat{a}^\dagger\hat{b}]$$
$$-\hbar\nu[\hat{a}^\dagger\hat{a}+\hat{b}^\dagger\hat{b}]-\mu\hbar[\alpha^*\hat{b}+\alpha\hat{b}^\dagger+\hat{a}^\dagger\beta+\hat{a}\beta^*]$$
$$+\lambda\hbar^2[(\alpha^*)^2\alpha^2+(\beta^*)^2\beta^2+2\alpha^*\alpha\beta^*\beta]$$
$$+\mu\hbar(\alpha^*\beta+\alpha\beta^*). \tag{41}$$

And the semiclassical time evolution operator is

$$\hat{U}_{\rm sc}(t)=\exp\left\{-i/\hbar\int_0^t\hat{H}_{\rm sc}(t')dt'\right\}$$
$$=\exp\left(-i\Omega\hat{a}^\dagger\hat{a}t-i\mu\int_0^t[\hat{a}^\dagger\beta(t')+\hat{a}\beta^*(t')]dt'\right)$$
$$\times\exp\left(-i\Omega\hat{b}^\dagger\hat{b}t-i\mu\int_0^t[\hat{b}^\dagger\alpha(t')+\hat{b}\alpha^*(t')]dt'\right)$$
$$=\hat{U}^a_{\rm sc}(t)\hat{U}^b_{\rm sc}(t). \tag{42}$$

By using some Lie algebraic technique and the parameters derivation technique we shall write

$$\hat{U}^a_{\rm sc}(t)=\exp\{i[\xi(t)\hat{a}^\dagger+\hat{a}\xi^*(t)]\}\exp(-i\Omega\hat{a}^\dagger\hat{a}t)$$
$$\times\exp\left(-i\Omega\int_0^t|\xi(t')|^2dt'\right), \tag{43}$$

$$\hat{U}^b_{\rm sc}(t)=\exp\{i[\rho(t)\hat{b}^\dagger+\hat{b}\rho^*(t)]\}\exp(-i\Omega\hat{b}^\dagger\hat{b}t)$$
$$\times\exp\left(-i\Omega\int_0^t|\rho(t')|^2dt'\right), \tag{44}$$

and we have

$$\phi(t)=\mu\int_0^t\beta(t')dt',$$

$$\psi(t)=\mu\int_0^t\alpha^*(t')dt'.$$

We now have the parameters $\xi(t)$ and $\rho(t)$ that obey the same differential equation as α and β , respectively. But the initial conditions are $\xi(0)=\rho(0)=0$.

We know that

$$\hat{U}^a_{\rm sc}(t)|\alpha_0\rangle=|\alpha_{\rm cls}(t)\rangle=\hat{\mathbf{D}}(\alpha_{\rm cls}(t))|0\rangle.$$

Then

$$\hat{U}^a_{\rm sc}(t)\hat{\mathbf{D}}(\alpha_0)=\hat{\mathbf{D}}(\alpha_{\rm cls}(t)).$$

$$\langle\hat{a}(t)\rangle=\alpha_{\rm cls}(t)\{1-2\lambda^2\hbar^2t^2[|\alpha(0)|^2+|\beta(0)|^2]\}+O(t^3).$$

The Ehrenfest time, for $\mu=0$, is

$$t_E\simeq[2\lambda^2\hbar^4(|\alpha(0)|^2+|\beta(0)|^2)]^{-1/2}=[2\lambda^2\hbar^2S_{\rm cls}]^{-1/2} \tag{45}$$

B. The linear entropy—quantum timescale and exact result

Now we will consider the following Hamiltonian:

$$\hat{H}_q=\hbar\tilde{\omega}(\hat{a}^\dagger\hat{a}+\hat{b}^\dagger\hat{b})+\lambda\hbar^2(\hat{a}^\dagger\hat{a}+\hat{b}^\dagger\hat{b})^2+\mu\hbar(\hat{a}^\dagger\hat{b}+\hat{a}\hat{b}^\dagger). \tag{46}$$

As we have $[(\hat{a}^\dagger\hat{a}+\hat{b}^\dagger\hat{b}),(\hat{a}^\dagger\hat{b}+\hat{a}\hat{b}^\dagger)]=0$, then we can write the time evolution operator as

$$\hat{\mathbf{U}}(t)=\exp\{-it\tilde{\omega}(\hat{a}^\dagger\hat{a}+\hat{b}^\dagger\hat{b})\}$$
$$\times\exp\{\lambda\hbar^2(\hat{a}^\dagger\hat{a}+\hat{b}^\dagger\hat{b})^2\}\exp\{\mu\hbar(\hat{a}^\dagger\hat{b}+\hat{a}\hat{b}^\dagger)\}.$$

Now we will study the time evolution of a two-dimensional coherent state under this Hamiltonian. The time evolved state is given by

$$|\Psi(t)\rangle=\exp\{-it\tilde{\omega}(\hat{a}^{\dagger}\hat{a}+\hat{b}^{\dagger}\hat{b})\}\exp\{-it\lambda\hbar(\hat{a}^{\dagger}\hat{a}+\hat{b}^{\dagger}\hat{b})^{2}\}\times\exp\{-it\mu(\hat{a}^{\dagger}\hat{b}+\hat{a}\hat{b}^{\dagger})\}|\alpha_{0},\beta_{0}\rangle.$$

But we know that

$$\exp\{-it\mu(\hat{a}^{\dagger}\hat{b}+\hat{a}\hat{b}^{\dagger})\}|\alpha_{0},\beta_{0}\rangle=|\tilde{\alpha}_{t},\tilde{\beta}_{t}\rangle,$$

where $\tilde{\alpha}_t=\alpha_0\cos(\mu t)-i\beta_0\sin(\mu t)$ and $\tilde{\beta}_t=-i\alpha_0\sin(\mu t)+\beta_0\cos(\mu t)$. Then with a few steps of algebraic calculations we may find

$$\begin{aligned}|\Psi(t)\rangle&=\exp\{-it\tilde{\omega}(\hat{a}^{\dagger}\hat{a}+\hat{b}^{\dagger}\hat{b})\}\times\exp\{-it\lambda\hbar(\hat{a}^{\dagger}\hat{a}+\hat{b}^{\dagger}\hat{b})^{2}\}|\tilde{\alpha}_{t},\tilde{\beta}_{t}\rangle\\&=e^{-|\widehat{\alpha_t}|^{2}/2}e^{-|\hat{\beta}_t|^{2}/2}\sum_{n,m}\exp\{-it\lambda\hbar(n+m)^{2}\}\times\frac{\hat{\beta}_t^{m}}{\sqrt{m!}}\frac{\hat{\alpha}_t^{n}}{\sqrt{n!}}|n\rangle|m\rangle,\end{aligned}\qquad(47)$$

where $\widetilde{\alpha_t}=\hat{\alpha}_t e^{-\tilde{\omega}}$ and $\hat{\beta}_t=\tilde{\beta}_t e^{-it\tilde{\omega}}$.

By making the product $|\Psi(t)\rangle\langle\Psi(t)|$, we find the following density operator:

$$\rho=e^{-|\hat{\alpha}_t|^{2}}e^{-|\hat{\beta}_t|^{2}}\sum_{n,m,p,q}\exp\{-it\lambda\hbar[(n+m)^{2}-(p+q)^{2}]\}\times\frac{\hat{\beta}_t^{m}}{\sqrt{m!}}\frac{\hat{\alpha}_t^{n}}{\sqrt{n!}}\frac{\hat{\beta}_t^{p}}{\sqrt{p!}}\frac{\hat{\alpha}_t^{q}}{\sqrt{q!}}|n\rangle\langle q|\otimes|m\rangle\langle p|.$$

The reduced density operator for the α liberty degree of freedom is

$$\rho_{\alpha}=e^{-|\hat{\alpha}_t|^{2}}e^{-|\hat{\beta}_t|^{2}}\sum_{p,m}\exp\{-it\lambda\hbar[m^{2}-p^{2}]\}\times\exp\{|\hat{\alpha}_t|^{2}e^{-2it\lambda\hbar(m-p)}\}\frac{\hat{\beta}_t^{m}}{\sqrt{m!}}\frac{\hat{\beta}_t^{p}}{\sqrt{p!}}|m\rangle\langle p|,$$

where $|m\rangle$ and $|p\rangle$ are Fock states. Then the square of the reduced density operator is

$$\rho_{\alpha}^{2}=e^{-2|\hat{\alpha}_t|^{2}}e^{-2|\hat{\beta}_t|^{2}}\sum_{m,q,n}\Bigg\{\exp\{-it\lambda\hbar[m^{2}-n^{2}]\}\times\exp\{|\hat{\alpha}_t|^{2}(e^{-2it\lambda\hbar(m-q)}+e^{-2it\lambda\hbar(q-n)})\}\times\frac{\hat{\beta}_t^{m}}{\sqrt{m!}}\frac{|\hat{\beta}_t|^{2q}}{q!}\frac{\hat{\beta}_t^{n}}{\sqrt{n!}}|m\rangle\langle n|\Bigg\}.\qquad(48)$$

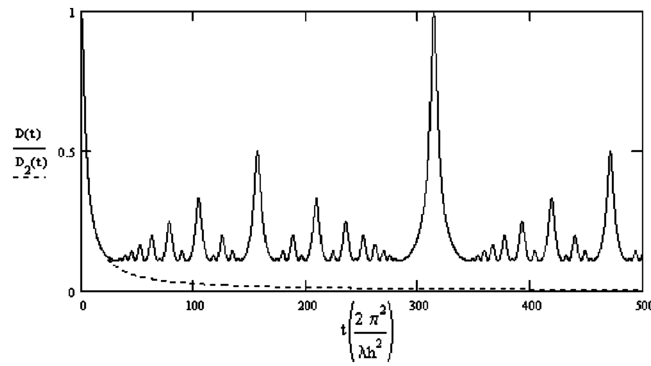

FIG. 3. The exact linear entropy (full line) and the short time linear entropy (dotted line) for the quartic two-dimensional harmonic oscillator.

The linear entropy is given by $\Delta_{\alpha}=1-\mathrm{Tr}(\rho_{\alpha}^{2})$ and then we need to evaluate the trace of ρ_{α}^{2}. By making some simple calculations we may find

$$\mathrm{Tr}[\rho_{\alpha}^{2}]=\sum_{q,n}e^{-2|\beta_0|^{2}}\exp(2|\alpha_0|^{2}\{\cos[2t\lambda\hbar^{2}(n-q)]-1\})\times\frac{|\hat{\beta}_t|^{2n}}{n!}\frac{|\hat{\beta}_t|^{2q}}{q!}.\qquad(49)$$

In order to obtain a typical time scale for this process we expand the cosine in the exponent and sum all leading contributions in t^{2}. The details of this laborious calculation are given in the Appendix. The result is

$$\mathrm{Tr}(\rho_{\alpha}^{2})\approx\frac{1}{\sqrt{1+|4S_{\mathrm{cls}}^{a}t^{2}\lambda^{2}S_{\mathrm{cls}}^{b}\hbar^{2}|}}.\qquad(50)$$

Note that the entanglement time scale is very different from what we called Ehrenfest's time scale,

$$t_{E}\propto\frac{1}{\sqrt{S_{\mathrm{cls}}^{a}+S_{\mathrm{cls}}^{b}}}\quad\text{and}\quad t_{D}\propto\frac{1}{\sqrt{S_{\mathrm{cls}}^{a}S_{\mathrm{cls}}^{b}}}$$

showing that if both actions S_{cls}^{a} and S_{cls}^{b} increase in the same manner, t_D is much shorter than t_E. The same result can be obtained by using Eq. (10) of Ref. [15]. In Fig. 3 we show the exact result for $\mathrm{Tr}(\rho_{\alpha}^{2})$ and compare with the short time behavior. If we look at the density expansion for short time, we immediately see that the first correction term brings a superposition of states and entanglement, i.e., the first term in the expansion makes use of the quantum kinematics.

IV. CLOSING REMARKS

In the present paper we set up a semiclassical approximation which helped us clarify, by means of several simple examples, the rich variety of time scales in the quantum domain. This is one of the reasons why constructing a bridge between classical and quantum mechanics is so difficult. The whole structure of both theories is completely different and several nonclassical phenomena should be attributed to the linear structure of the Hilbert space, a feature which is absent in the classical context. Application of this present development in chaotic situations is in order.

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APPENDIX

First we set $\mu=0$, then we get $|\hat{\alpha}_t|^2=|\alpha_0|^2$. We want to extract the $\hbar$ independent contribution to $\mathrm{Tr}(\rho_\alpha^2)$. For this purpose it suffices to sum all contribution coming from the second order of the expansion of the cosine function. Then for the first order we find $\Delta^0=0$ then we have:

$$\mathrm{Tr}[\rho_\alpha^2]=\sum_{q,n} e^{-2|\beta_0|^2}\exp\left\{2|\alpha_0|^2\sum_{m=0}\frac{[2t\lambda\hbar^2(n-q)]_{2m+2}}{(2m+2)!}\right\}\times\frac{|\beta_0|^{2n}}{n!}\frac{|\beta_0|^{2q}}{q!}$$

$$\simeq\sum_{q,n} e^{-2|\beta_0|^2}\exp[-4|\alpha_0|^2t^2\lambda^2\hbar^4(n-q)^2]\times\frac{|\beta_0|^{2n}}{n!}\frac{|\beta_0|^{2q}}{q!}.$$

Let us examinee the following function:

$$f(x,y)=\sum_{n,p}\frac{(xa)^n(ya)^p}{n!p!}=e^{(x+y)a}.$$

If we apply once the differential operator $(x\partial/\partial x-y\partial/\partial y)$ we find

$$\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)f(x,y)=a(x-y)f(x,y).$$

Then we can see that

$$\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)f(x,y)=\sum_{n,p}\frac{(n-p)(xa)^n(ya)^p}{n!p!}=a(x-y)e^{(x+y)a}.$$

So for $a=|\beta_0|^2$ we get

$$\sum_{n,p}\frac{(n-p)^2(|\beta_0|^2)^n(|\beta_0|^2)^p}{n!p!}=2|\beta_0|^2e^{2|\beta_0|^2}.$$

Let

$$D=\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right),$$

$$a=|\beta_0|^2,$$

$$s=x+y,$$

$$d=x-y,$$

we have $Ds=d$ and $Dd=s$, and also

$$De^{as}=ade^{as},$$

$$D(g(s,d)e^{as}(=[adg(s,d)+D(g(s,d))]e^{as}.$$

The general result is then $D^ne^{as}=g^n(s,d)e^{as}$, with

$$g^0(s,d)=1,$$

$$g^1(s,d)=ad,$$

$$g^2(s,d)=a^2d^2+as,$$

$$g^3(s,d)=a^3d^3+3a^2ds+ad,$$

$$g^3(s,0)=0,$$

$$g^4(s,d)=a^4d^4+6a^3d^2s+4a^2d^2+3a^2s^2+as,$$

$$\vdots$$

If we only consider the smallest term in power of $\hbar$ at each application of the operator D, we can obtain an approximated linear entropy. As we will see, this will give us an $\hbar$ independent linear entropy. By doing this we have

$$g^4(s,d)\approx a^4d^4+6a^3d^2s+3a^2s^2,$$

$$g^5(s,d)\approx a^5d^5+10a^4d^3s+15a^3ds^2,$$

$$g^6(s,d)\approx a^6d^6+15a^5d^4s+45a^4d^2s^2+15a^3s^3,$$

$$g^7(s,d)\approx a^7d^7+21a^6d^5s+105a^5d^3s^2+105a^4s^3d,$$

$$g^8(s,d)\approx a^8d^8+28a^7d^6s+210a^6d^4s^2+420a^5s^3d^2+105a^4s^4$$

For $x=y=1$ we can see that

$$g^0(s,d)=1=0!!(2a)^0,$$

$$g^2(2,0)=2a=1!!(2a)^1,$$

$$g^4(2,0)\approx 3(2a)^2=3!!(2a)^2,$$

$$g^6(2,0)\approx 5!!(2a)^3,$$

$$g^8(2,0)\approx 7!!(2a)^4.$$

Then we find

$$\sum_{n,p}\frac{(n-p)^{2m}(|\beta_0|^2)^n(|\beta_0|^2)^p}{n!p!}\approx(2m-1)!!(2|\beta_0|^2)^m e^{2|\beta_0|^2}.$$

For the trace we get

$$\mathrm{Tr}(\rho_\alpha^2)\approx\sum_{q,n} e^{-2|\beta_0|^2}\exp[-4|\hat{\alpha}_0|^2t^2\lambda^2\hbar^4(n-q)^2]\times\frac{|\beta_0|^{2n}}{n!}\frac{|\beta_0|^{2q}}{q!}\approx 1+\sum_{k=1}^{\infty}\frac{(2k-1)!!}{k!}(-2|\hat{\alpha}_0t\lambda\hbar^2\beta_0|^2)^k.$$

As $(1+u)^{-1/2}=1-1!!u/1!2+3!!/2!(u/2)^2+\cdots$, then finally we get

$$\mathrm{Tr}(\rho_\alpha^2)\approx\frac{1}{\sqrt{1+4|\alpha_0t\lambda\hbar^2\beta_0|^2}}=\frac{1}{\sqrt{1+4|S^a_{\mathrm{cls}}t^2\hbar^2\lambda^2S^b_{\mathrm{cls}}|}} \tag{A1}$$

[1] L.E. Ballentine, Y. Yang, and J.P. Zibim, Phys. Rev. A **50**, 2854 (1994).
[2] L. Ballentine and S. McRac, Phys. Rev. A **58**, 1799 (1998).
[3] L. Ballentine, Phys. Rev. A **63**, 31 (2001).
[4] K.M. Fonseca Romero, M.C. Nemes, J.G. Peixoto de Faria, and A.F.R. de Toleda Piza, e-print quant-ph/0201072.
[5] E. Schrödinger, Naturwissenschaften **14**, 644 (1926).
[6] F. Zhang and Gilmore, Rev. Mod. Phys. **62**, 867 (1990).
[7] G. Berman and G. Zaslavsky, Physica A **91**, 450 (1977).
[8] A.G.P. Berman and G. Zaslavsky, Physica D **4**, 113 (1981).
[9] R.A.L. Sanz and K. Furuya, e-print quant-ph/0210162.
[10] K. Kheruntsyan, J. Opt. B: Quantum Semiclassical Opt. **1**, 225 (1999).
[11] R. Angelo, Ph.D. thesis, UNICAMP, 2003.
[12] T.W.H.M. Greiner, O. Mandel and I. Bloch, Nature (London) **419**, 51 (2002).
[13] Adélcio C. Oliveira and M. C. Nemes (unpublished): we have numerically evaluated the expectation value of $\hat{x}$ and $\hat{p}$ according to a Liouvillian dynamics. We note that deviations from the corresponding quantum behavior are essentialy due typical quantum effects such as dephasing in time evolution and revivals, which are completely absent from the classical description. The presented Ehrenfest time Eq. (23) contains essentially the effect of wave-packet distortion which is present in both descriptions if we artificially start with a classical wave packet whose width depends on $\hbar$.
[14] D.F. Walls and G.J. Milburn, *Quantum Optics* (Springer-Verlag, Berlin, 1995).
[15] Ji Il Kim, M.C. Nemes, A. de Toledo Pizza, and H. Borges, Phys. Rev. Lett. **77**, 207 (1996).